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Continuum Sizing Design Sensitivity Analysis of Eigenvectors Using Ritz Vectors

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I. Introduction

DESIGN sensitivity analysis (DSA) of eigenvectors is important for the study of structural dynamics, e.g., to identify and optimize structural systems, and substantial efforts have been made in this area.[1,2] Recently, Wang[3,4] proposed two modified modal methods, called the explicit and implicit methods. In the explicit method, the first load-dependent Ritz vector (LDRV)[5–7] is used in the solution, while in the implicit method, the first LDRV is added to the basis vectors. Wang[4] numerically demonstrated that the implicit method is superior to the explicit method. Nevertheless, all discrete methods of DSA[1–4] require derivatives of stiffness and mass matrices.

This Note proposes a unified continuum-based sizing DSA of eigenvectors that can be obtained without differentiating stiffness and mass matrices. Fox and Kapoor's eigenvector expansion method is modified by adding LDRVs to the basis to improve the accuracy of design sensitivity of eigenvectors. LDRVs proposed by Kline[6,7] are used for a numerical example. The numerical example studied in this Note shows that adding two LDRVs to the basis vectors yields accurate sensitivity results.

In this Note, uppercase bold letters denote matrices, lowercase bold italic letters are vectors, and lowercase letters are either scalars or functions.

II. Continuum Design Sensitivity of Eigenvectors

In order to avoid differentiation of stiffness and mass matrices, the variational equations of eigenvalue problems are differentiated in the continuum design sensitivity of eigenfunctions. For a given structural system with physical domain Ω, variational equations of eigenvalue problems can be written as[8]

$$a_u(y^i, \bar{y}) = \zeta_i d_u(y^i, \bar{y}) \tag{1}$$

for all $\bar{y} \in Z$; and the orthonormalizing condition employed is

$$d_u(y^i, y^j) = d_u(\psi^i, \psi^j) = d_u(\phi^i, \phi^j) = \delta_{ij} \tag{2}$$

where Z is the vector space of kinematically admissible displacements, $a_u(\ )$ is the strain energy bilinear form, $d_u(\ )$ is the mass effect bilinear form, $\bar{y}(\boldsymbol{x})$ is the kinematically admissible eigenfunction with $\boldsymbol{x} \in \Omega$, $y(\boldsymbol{x})$ is the eigenfunction, ζ is the eigenvalue, $\psi(\boldsymbol{x})$ is the Ritz function, ϕ is the basis function, i.e., either the eigenfunction or Ritz function, and δ_{ij} is the Kronecker delta. In Eqs. (1) and (2), the subscript u denotes dependency of the bilinear forms on a design variable u.

Design sensitivity of the eigenvalue is completed first. Substituting $\bar{y} = y^j$ in Eq. (1) and using Eq. (2)

$$a_u(y^i, y^j) = \zeta_i \delta_{ij} \tag{3}$$

Taking the first variation of Eq. (1) with respect to a design variable u yields

$$\begin{aligned} a'_{\delta u}(y^i, \bar{y}) + a_u(y^{i'}, \bar{y}) &= \zeta'_i d_u(y^i, \bar{y}) \\ &+ \zeta_i d'_{\delta u}(y^i, \bar{y}) + \zeta_i d_u(y^{i'}, \bar{y}) \end{aligned} \tag{4}$$

which must hold for all $\bar{y} \in Z$. Letting $\bar{y} = y^i$ in Eq. (4) and using Eq. (2)

$$\begin{aligned} \zeta'_i = a'_{\delta u}(y^i, y^i) - \zeta_i d'_{\delta u}(y^i, y^i) + [a_u(y^{i'}, y^i) \\ - \zeta_i d_u(y^{i'}, y^i)] \end{aligned} \tag{5}$$

Since $y^{i'}$ satisfies the same kinematic boundary conditions as y^i, thus $y^{i'} \in Z$.[8] Using Eq. (1), the last two terms of Eq. (5) cancel each other, and design sensitivity of the ith eigenvalue ζ_i is

$$\zeta'_i = a'_{\delta u}(y^i, y^i) - \zeta_i d'_{\delta u}(y^i, y^i) \tag{6}$$

Rearranging Eq. (4), the continuum equation for the design sensitivity of the ith eigenfunction y^i is obtained as

$$\begin{aligned} a_u(y^{i'}, \bar{y}) - \zeta_i d_u(y^{i'}, \bar{y}) &= -a'_{\delta u}(y^i, \bar{y}) \\ &+ \zeta_i d'_{\delta u}(y^i, \bar{y}) + \zeta'_i d_u(y^i, \bar{y}) \end{aligned} \tag{7}$$

which must hold for all $\bar{y} \in Z$.

III. Approximate Design Sensitivity Analysis

The design sensitivity of the ith eigenfunction y^i with respect to a design variable u is approximated by

$$y^{i'} = \boldsymbol{\phi c} \tag{8}$$

where $\boldsymbol{\phi}$ is a vector function such that $\boldsymbol{\phi} = [y(\boldsymbol{x})^T, \boldsymbol{\psi}(\boldsymbol{x})^T]^T = [y^1(\boldsymbol{x}) \ldots y^q(\boldsymbol{x}), \psi^1(\boldsymbol{x}) \ldots \psi^{r-q}(\boldsymbol{x})]^T$. Substituting Eq. (8) into Eq. (7) and letting $\bar{y} = \phi^j$, $k = 1, 2, \ldots, r$, in Eq. (7), and $r \times r$ matrix equation is obtained as

$$\begin{aligned} [a_u(\phi^k, \phi^j) - \zeta_i d_u(\phi^k, \phi^j)]\boldsymbol{c} = [-a'_{\delta u}(y^i, \phi^j) \\ + \zeta_i d'_{\delta u}(y^i, \phi^j) + \zeta'_i d_u(y^i, \phi^j)] \end{aligned} \tag{9}$$

Presented as Paper 91-1092 at the AIAA/ASME/ASCE/AHS/ASC 32nd Structures, Structural Dynamics and Materials Conference, Baltimore, MD, April 8–10, 1991; received Dec. 14, 1991; revision received Feb. 17, 1993; accepted for publication Feb. 19, 1993.

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Equation (9) can be partitioned as

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{12}^T & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} c^1 \\ c^2 \end{bmatrix} = \begin{bmatrix} d^1 \\ d^2 \end{bmatrix} \tag{10}$$

where

$\mathbf{A}_{11}$ = $q \times q$ diagonal matrix with diagonal terms $\zeta_j - \zeta_i$, $j = 1$ to q
$\mathbf{A}_{12}$ = $q \times (r - q)$ zero matrix
$\mathbf{A}_{22}$ = $(r - q) \times (r - q)$ nonsingular full matrix $a_u(\phi^k, \phi^j) - \zeta_i \delta_{kj}$
$\boldsymbol{d}^1$ = $q \times 1$ vector $- a'_{\delta u}(y^i, y^j) + \zeta_i d'_{\delta u}(y^i, y^j) + \zeta'_i \delta_{ij}$, $j = 1$ to q
$\boldsymbol{d}^2$ = $(r - q) \times 1$ vector $- a'_{\delta u}(y^i, \psi^{k-q}) + \zeta_i d'_{\delta u}(y^i, \psi^{k-q})$, $k = q + 1$ to r

Since $\mathbf{A}_{11}$ is diagonal, Eq. (10) yields

$$c_j = \frac{-a'_{\delta u}(y^i, y^j) + \zeta_i d'_{\delta u}(y^i, y^j)}{\zeta_j - \zeta_i}, \qquad j \neq i, \quad j = 1 \text{ to } q \tag{11}$$

and c_k, $k = q + 1$ to r, can be obtained using a matrix equation solver.

Differentiating the normalizing condition, $d_u(y^i, y^i) = 1$, with respect to a design variable u yields

$$2d_u(y^{i'}, y^i) + d'_{\delta u}(y^i, y^i) = 0 \tag{12}$$

Equation (8) can be rewritten as

$$y^{i'} = \sum_{\substack{j=1 \\ j \neq i}}^{q} c_j y^j + c_i y^i + \sum_{k=q+1}^{r} c_k \psi^{k-q} \tag{13}$$

Substituting Eq. (13) into Eq. (12), and using Eq. (2)

$$c_i = -\tfrac{1}{2} d'_{\delta u}(y^i, y^i) \tag{14}$$

Thus, the continuum-based design sensitivity of the ith eigenfunction y^i with respect to u is

$$y^{i'} = \sum_{\substack{j=1 \\ j \neq i}}^{q} \frac{-a'_{\delta u}(y^i, y^j) + \zeta_i d'_{\delta u}(y^i, y^j)}{\zeta_j - \zeta_i} y^j - \frac{1}{2} d'_{\delta u}(y^i, y^i) y^i + \sum_{k=q+1}^{r} c_k \psi^{k-q} \tag{15}$$

Once the discretized eigenvector $\boldsymbol{y}^i$ and LDRV $\boldsymbol{\psi}^k$ are obtained using the finite element method, Eq. (15) can be discretized using shape functions as

$$\boldsymbol{y}^{i'} = \sum_{\substack{j=1 \\ j \neq i}}^{q} \frac{-a'_{\delta u}(y^i, y^j) + \zeta_i d'_{\delta u}(y^i, y^j)}{\zeta_j - \zeta_i} \boldsymbol{y}^j - \frac{1}{2} d'_{\delta u}(y^i, y^i) \boldsymbol{y}^i + \sum_{k=q+1}^{r} c_k \boldsymbol{\psi}^{k-q} \tag{16}$$

It is important to note that if the continuum method of DSA is used, the design sensitivity expression of Eq. (16) does not require differentiation of stiffness and mass matrices. Design sensitivity can thus be implemented for established finite element codes by postprocessing finite element analysis results.

The first variations of the energy bilinear forms of Eq. (16) for the truss/beam and plane elastic solid/plate are given in Ref. 5, and numerical computation of the bilinear form for the truss design component are given in the Appendix.

IV. Comparison of Methods

In this section, four modal approximation methods are compared: 1) Fox and Kapoor's[1] eigenvector expansion method, 2) the explicit method, 3) the implicit method, and 4) the continuum-based method.

The four methods for computing design sensitivity of eigenvectors are presented in Table 1. The explicit method reduces the truncation error of Fox and Kapoor's method by adding the first LDRV in the solution. The implicit method, which adds the first LDRV to the basis, further reduces the error by multiplying the correction factor.[9] Finally, the continuum-based method improves the accuracy of the implicit method by using more than one LDRV; i.e., the second LDRV contains both stiffness and inertia terms, whereas the first LDRV contains the stiffness term only. Numerical results in the next section show that the continuum-based method gives the most accurate results. Moreover, the continuum-based method is more efficient since it does not require derivatives of matrices.

V. Numerical Example

A 108-member helicopter tail boom is studied.[5] ANSYS is used to obtain stiffness and mass matrices, and eigenpairs. A FORTRAN program is written to generate LDRVs. Various basis vectors are used to obtain the design sensitivity of the fifth eigenvector with respect to the third design variable. Exact design sensitivity of eigenvectors is denoted by y'_E, which is obtained by using all eigenvectors, and y' is the approximate design sensitivity. The ratio between y' and y'_E times 100 is used as a measure of the accuracy of the design sensitivity computation. For brevity, Table 2 shows only the first 10 components of the design sensitivity of the eigenvector.

Table 2 shows that the accuracy of sensitivity is improved by adding one LDRV to five eigenvectors. This basis is better than 30 eigenvectors. However, adding two LDRVs to five

Table 1 Comparison of four methods

Fox and Kapoor's method	$\boldsymbol{y}^i_{,p} = \sum_{\substack{j=1 \\ j \neq i}}^{q} \frac{-\boldsymbol{y}^{j^T}(\mathbf{K}_{,p} - \zeta_i \mathbf{M}_{,p})\boldsymbol{y}^i}{\zeta_j - \zeta_i} \boldsymbol{y}^j - \frac{1}{2} (\boldsymbol{y}^{i^T} \mathbf{M}_{,p} \boldsymbol{y}^i) \boldsymbol{y}^i$
Explicit method	$\boldsymbol{y}^i_{,p} = \sum_{\substack{j=1 \\ j \neq i}}^{q} \frac{-\boldsymbol{y}^{j^T}(\mathbf{K}_{,p} - \zeta_i \mathbf{M}_{,p})\boldsymbol{y}^i}{\zeta_j - \zeta_i} \boldsymbol{y}^j - \frac{1}{2} (\boldsymbol{y}^{i^T} \mathbf{M}_{,p} \boldsymbol{y}^i) \boldsymbol{y}^i + \boldsymbol{\psi}^1$
Implicit method	$\boldsymbol{y}^i_{,p} = \sum_{\substack{j=1 \\ j \neq i}}^{q} \frac{-\boldsymbol{y}^{j^T}(\mathbf{K}_{,p} - \zeta_i \mathbf{M}_{,p})\boldsymbol{y}^i}{\zeta_j - \zeta_i} \boldsymbol{y}^j - \frac{1}{2} (\boldsymbol{y}^{i^T} \mathbf{M}_{,p} \boldsymbol{y}^i) \boldsymbol{y}^i + \gamma \boldsymbol{\psi}^1$
Continuum-based method	$\boldsymbol{y}^{i'} = \sum_{\substack{j=1 \\ j \neq i}}^{q} \frac{-a'_{\delta u}(y^i, y^j) - \zeta_i d'_{\delta u}(y^i, y^j)}{\zeta_j - \zeta_i} \boldsymbol{y}^j - \frac{1}{2} d'_{\delta u}(y^i, y^i) \boldsymbol{y}^i + \sum_{k=q+1}^{r} c_k \boldsymbol{\psi}^{k-q}$

Table 2 Design sensitivity of the fifth eigenvector of a tail boom truss with respect to the third design variable where design sensitivity of the fifth eigenvalue with respect to the third design variable is −19,479.0

		30 EVs		5 EVs and 1 LDRV		5 EVs and 2 LDRVs		5 EVs and 3 LDRVs	
5th Eigenvector	y'_E	y'	$\frac{y'}{y'_E}$, %	y'	$\frac{y'}{y'_E}$, %	y'	$\frac{y'}{y'_E}$, %	y'	$\frac{y'}{y'_E}$, %
−0.4732D−1	−0.1400D−1	−0.6790D−2	48.5	−0.144D−1	102.5	−0.130D−1	99.3	−0.141D−1	100.1
0.1111D−1	0.4264D−1	0.4269D−1	100.1	0.4796D−1	112.5	0.4294D−1	100.7	0.4269D−1	100.1
−0.1880D+1	−0.7757D−1	−0.7816D−1	100.8	−0.6976D−1	89.9	−0.7785D−1	100.4	−0.7764D−1	100.1
−0.4750D−1	−0.1344D−1	−0.5631D−2	41.9	−0.1383D−1	102.9	−0.1333D−1	99.2	−0.1344D−1	100.0
−0.1525D−1	−0.9496D−1	−0.9496D−1	100.0	−0.9838D−1	103.6	−0.9488D−1	99.9	−0.9489D−1	99.9
−0.1886D+1	−0.7971D−1	−0.8036D−1	100.8	−0.6962D−1	87.3	−0.7959D−1	99.8	−0.7966D−1	99.9
0.4748D−1	0.1405D−1	0.6234D−2	44.4	0.1446D−1	102.9	0.1394D−1	99.2	0.1405D−1	100.0
0.1459D−1	0.4950D−1	0.4952D−1	100.0	0.5253D−1	106.1	0.4940D−1	99.8	0.4943D−1	99.9
−0.1885D+1	−0.7801D−1	−0.7858D−1	100.7	−0.6789D−1	87.0	−0.7788D−1	99.8	−0.7796D−1	99.9
0.4735D−1	0.1275D−1	0.5903D−2	46.3	0.1302D−1	102.1	0.1265D−1	99.2	0.1275D−1	100.0

Note: EV stands for eigenvector.

eigenvectors gives excellent results. Adding three LDRVs, nevertheless, does not yield a significant improvement.

VI. Conclusion

A unified continuum-based sizing DSA of eigenvectors using LDRVs has been proposed. This method does not require derivatives of stiffness and mass matrices. Furthermore, use of LDRVs makes this method inexpensive and improves the accuracy and convergence of sensitivity. A helicopter tail boom truss is studied and very good sensitivity results for eigenvectors are obtained by adding two LDRVs to the existing eigenvector basis.

Appendix

Energy bilinear forms for the truss element can be written as

$$a_u(y, \bar{y}) = \int_0^l AEy_{,x}\bar{y}_{,x}\,dx, \qquad d_u(y, \bar{y}) = \int_0^l A\rho y\bar{y}\,dx \tag{A1}$$

where y and $\bar{y}$ are the eigenfunction and virtual eigenfunction, respectively, A is the cross-section area, E is Young's modulus, ρ is density, and $(\ )_{,x}$ is the derivative with respect to x. The first design variation of the bilinear form $a_u(y, \bar{y})$ with respect to the cross-sectional area A is

$$a'_{\delta u}(y, \bar{y}) = \int_0^l Ey_{,x}\bar{y}_{,x}\delta A\,dx = E\int_0^l (N_{a,x}y)(N_{a,x}\bar{y})\delta A\,dx \tag{A2}$$

where the shape function is

$$N_a = \left[\frac{l-x}{l}, 0, 0, \frac{x}{l}, 0, 0\right] \tag{A3}$$

In the same way, $d'_{\delta u}(y, \bar{y})$ can be obtained.

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Scheduled Maintenance Optimization System

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Introduction

THE cost of scheduled maintenance is significant. However, cost is not always considered in the up-front design stages. Weapon systems require many scheduled maintenance and ground support activities to insure safe and successful missions. Scheduled maintenance is needed to assure meeting mission requirements, and at the same time can restrict maximum sortie generations due to the downtime required to perform inspection and ground handling tasks.

The MDA maintainability attainment independent research and development (IRAD) no. 7-925 studied methods of reducing Air Force and Navy aircraft turnaround and scheduled maintenance requirements. The results of the IRAD study were provided during a symposium at Warner Robins AFB in 1990.[1] A data base management system (DBMS) was developed to analyze and study both Air Force and Navy air-

Received Jan. 5, 1993; revision received March 8, 1993; accepted for publication March 9, 1993.

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